

# Non-Gaussianity in Nonminimally Coupled Scalar Field Theory

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## Abstract

We consider the non-Gaussianity of the nonlinear density perturbations in a single-field inflationary model when a scalar field couples nonminimally with gravity. Gravity theories with a nonminimal coupling can be transformed into the Einstein gravity with canonical kinetic terms by using a suitable conformal transformation. We find that a nonlinear generalization of the gauge invariant quantity  $\zeta_i$  is invariant under the conformal transformation. With the help of this conformal invariant property, we calculate the non-Gaussianity, which is characterized by a nonlinear parameter  $f_{NL}$ , in nonminimal coupled scalar field theory.

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## I. INTRODUCTION

The inflation scenario can explain successfully a seed for density perturbations which are adiabatic and Gaussian, as well as the problems of Friedmann-Robertson-Walker(FRW) cosmology, such as the flatness and the horizon problems [1, 2]. Also observational data fit the theoretical predictions within linear perturbation theory to a good accuracy [3, 4, 5]. However, recent observations with high precision (*e.g.*, WMAP, PLANCK) [6] require more accurate perturbation theory, beyond linear order, during the inflation period. For example, if non-linear perturbations are taken into account, a non-Gaussian signal is expected to be detected on the temperature anisotropy in future experiments [7, 8]. Detection of the non-Gaussianity can give information about the generation mechanism of density perturbations (such as inflaton-, curvaton-, or inhomogeneous reheating scenario) [9] and discriminate inflation models (such as single-field [7, 8, 10] or multi-field inflation [11]).

Generalized gravity theories beyond the Einstein gravity naturally emerge from the fundamental physics theory, such as superstring/M-theory, and are thought to explain the present accelerating universe. Non-Gaussianity in generalized gravity theory can also be calculated using Hamilton-Jacobi formalism in Ref. [12]. In this paper, we calculate the non-Gaussianity during inflation when the scalar field is coupled to the gravitational curvature nonminimally. Instead of the Hamilton-Jacobi approach in generalized gravity theory, which was adopted in Ref. [12], we use the conformal transformation method that transforms nonminimally coupled scalar field theory into the canonical Einstein gravity. Further, we find that the nonlinear curvature perturbation quantity on uniform field hypersurfaces (comoving hypersurfaces) is invariant under the conformal transformation.

## II. CURVATURE PERTURBATIONS IN NONMINIMALLY COUPLED SCALAR FIELD THEORY

We start with the action for which the scalar field is coupled to the Ricci scalar nonminimally:

$$S = \int d^4x \sqrt{-g} \left[ (1 - \kappa \xi \phi^2) \frac{R}{2\kappa} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad (1)$$

where  $\xi$  is a nonminimal coupling constant and  $\kappa = 8\pi G$ . We consider the Arnowitt-Deser-Misner (ADM) metric

$$ds^2 = -N^2 dt^2 + \gamma_{ij} dx^i dx^j, \quad (2)$$

where  $N$  and  $\gamma_{ij}$  are a lapse function and a 3-spatial metric, respectively. The 3-spatial metric  $\gamma_{ij}$  can be written as

$$\gamma_{ij} = a^2(t) e^{2\alpha(t, \mathbf{x})} \delta_{ij}. \quad (3)$$

In this paper, we neglect the gravitational wave contribution [12, 13]. The local Hubble parameter  $H$  takes the form

$$H \equiv \frac{\dot{a}}{Na} + \frac{\dot{\alpha}}{N}. \quad (4)$$

The gauge invariant quantity in nonlinear theory was introduced in the literature [12, 13, 14] and is given by

$$\zeta_i = \partial_i \alpha - \frac{NH}{\dot{\phi}} \partial_i \phi. \quad (5)$$

If we expand  $\zeta_i$  up to second order as  $g = g_1 + \frac{1}{2}g_2$ , then the linear and the second order gauge invariant quantities are obtained in single field inflation model [7, 15] as

$$\zeta_1 = \alpha_1 - \frac{H_0}{\dot{\phi}_0} \delta\phi, \quad (6)$$

$$\begin{aligned} \zeta_2 = & \alpha_2 - \frac{H_0}{\dot{\phi}_0} \phi_2 + 2 \frac{H_0}{\dot{\phi}_0^2} \phi_1 \dot{\phi}_1 - 2 \frac{\phi_1}{\dot{\phi}_0} \dot{\alpha}_1 \\ & + \frac{H_0 \phi_1^2}{\dot{\phi}_0^2} \left( \frac{\dot{H}_0}{H_0} - \frac{\ddot{\phi}_0}{\dot{\phi}_0} \right), \end{aligned} \quad (7)$$

where  $\zeta_i = \partial_i \zeta$ . Similarly to linear perturbation theory,  $\zeta_i$  can be understood as a curvature perturbation on comoving hypersurfaces (or uniform field hypersurfaces) and is conserved when the perturbation length scales are larger than the horizon scale for an adiabatic perturbation.

Nonlinear perturbations may generate a non-Gaussian signal on the temperature anisotropy. To show this non-Gaussianity,  $\zeta$  can be decomposed into linear and nonlinear parts with a nonlinear parameter  $f_{NL}$  [16]:

$$\zeta = \zeta_L + \frac{3}{5} f_{NL} (\zeta_L^2 - \langle \zeta_L^2 \rangle), \quad (8)$$

where  $\zeta_L$  is a linear Gaussian perturbation that satisfies  $\langle \zeta_L \rangle = 0$ . With this definition, the non-vanishing component of the  $\zeta(\mathbf{x})$  bispectrum is

$$\begin{aligned} \langle \zeta_L(\mathbf{k}_1) \zeta_L(\mathbf{k}_2) \zeta_{NL}(\mathbf{k}_3) \rangle &= 2(2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \\ &\times f_{NL} P_\zeta(\mathbf{k}_1) P_\zeta(\mathbf{k}_2), \end{aligned} \quad (9)$$

where

$$\langle \zeta_L(\mathbf{k}_1) \zeta_L(\mathbf{k}_2) \rangle = (2\pi)^3 P_\zeta(\mathbf{k}_1) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2), \quad (10)$$

$$\begin{aligned} \zeta_{NL}(\mathbf{k}) &= f_{NL} \left[ \int \frac{d^3 k'}{(2\pi)^3} \zeta_L(\mathbf{k} + \mathbf{k}') \zeta_L^*(\mathbf{k}') \right. \\ &\quad \left. - (2\pi)^3 \delta^{(3)}(\mathbf{k}) \langle \zeta_L^2(\mathbf{x}) \rangle \right]. \end{aligned} \quad (11)$$

WMAP data give an observational constraint on  $f_{NL}$  such that  $-58 < f_{NL} < 134$  [6].

### III. CONFORMAL TRANSFORMATION

It is more convenient to deal with the equations of motion in the Einstein gravity with a minimally coupled scalar field by using a suitable conformal transformation

$$\hat{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu}. \quad (12)$$

Applying the conformal transformation with  $\Omega^2 = 1 - \kappa\xi\phi^2$  and introducing a new scalar field  $\hat{\phi}$  to make a canonical Lagrangian form as

$$\hat{\phi} = \int d\phi F(\phi), \quad (13)$$

we obtain the action in the Einstein frame:

$$S = \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{2\kappa} \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - \hat{V}(\hat{\phi}) \right], \quad (14)$$

where

$$F(\phi) = \frac{\sqrt{1 - \kappa\xi(1 - 6\xi)\phi^2}}{\Omega^2}, \quad \hat{V}(\hat{\phi}) = \frac{V(\phi)}{\Omega^4}. \quad (15)$$

The  $\zeta$  in linear perturbation theory has been verified to be invariant under a conformal transformation in inflation theory with a nonminimal coupling [17] and in generalized gravity cases [18]. The conformal invariance of  $\zeta$  implies that the power spectrum of  $\zeta$  is the same

in both frames. It is important to check whether the nonlinear generalization of  $\zeta$ - $\zeta_i$  is also invariant under the conformal transformation. To do so, we decompose the conformal factor  $\Omega$  into a homogeneous part, which depends only on the time, and inhomogeneous part as

$$\Omega(t, \mathbf{x}) = \Omega_0(t)e^{\omega(t, \mathbf{x})}. \quad (16)$$

Under this decomposition, we can easily check that the quantities in each frame are related as

$$\begin{aligned} d\hat{t} &= \Omega_0 dt, & \hat{a} &= \Omega_0 a, \\ \hat{N} &= e^\omega N, & \hat{\alpha} &= \alpha + \omega, \\ \hat{H} &= \frac{1}{\Omega} \left( H + \frac{\dot{\Omega}}{N\Omega} \right). \end{aligned} \quad (17)$$

With these relations, we find

$$\hat{\zeta}_i = \partial_i \hat{\alpha} - \frac{\hat{N}\hat{H}}{\dot{\hat{\phi}}} \partial_i \hat{\phi} \quad (18)$$

$$= \partial_i \alpha - \frac{NH}{\dot{\phi}} \partial_i \phi = \zeta_i. \quad (19)$$

The nonlinear gauge invariant quantity  $\zeta_i$  in the original frame exactly coincides with that in the transformed Einstein frame to nonlinear order. Therefore, the power spectrum and the bispectrum of  $\zeta_i$  also coincide in each frame:

$$\hat{P}_\zeta = P_\zeta, \quad \langle \hat{\zeta}_{k_1} \hat{\zeta}_{k_2} \hat{\zeta}_{k_3} \rangle = \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle. \quad (20)$$

From the definition of  $f_{NL}$  in Eq. (8), conformal invariance of  $\zeta_i$  gives  $\hat{f}_{NL} = f_{NL}$ .

#### IV. THREE-POINT CORRELATION FUNCTIONS AND NON-GAUSSIANITY

While the nonvacuum initial states can modify a four-point correlation compared to the Gaussian statistics [19], nonlinear perturbations give a nonzero three-point correlation that vanishes in Gaussian statistics. Since the bispectrum in the original Jordan frame is the same as that in the Einstein frame, as shown in Sect. III, we may borrow the results of Maldacena [8]. The third order action in terms of  $\zeta$  has been constructed in Ref. [8] to calculate the bispectrum in the Einstein frame to the  $\epsilon^2$  order by the solving constraint

equations and is given as

$$S_3 = \int d^4x \left[ \left( \frac{4\pi}{m_{pl}} \right)^2 \frac{\dot{\phi}^4}{\hat{H}^4} (\hat{a}^3 \dot{\zeta}^2 \hat{\zeta} + \hat{a} (\partial \hat{\zeta})^2 \hat{\zeta}) - \frac{4\pi}{m_{pl}} \frac{\dot{\phi}^2}{\hat{H}^2} \dot{\zeta} (\partial \hat{\zeta} \cdot \partial \hat{\chi}) \right] = \int d^4\mathcal{L}_I, \quad (21)$$

where  $\partial^2 \hat{\chi} = \frac{\dot{\phi}^2}{2\hat{H}^2} \dot{\zeta}$ . The slow-roll parameters  $\hat{\epsilon}$  and  $\hat{\eta}$  are defined by

$$\hat{\epsilon} = -\frac{\dot{\hat{H}}}{\hat{H}^2} = -\frac{4\pi}{m_{pl}^2} \frac{\dot{\phi}^2}{\hat{H}^2} = \frac{m_{pl}^2}{16\pi} \left( \frac{\hat{V}_{,\phi}}{\hat{V}} \right)^2 \equiv \hat{\epsilon}_V, \quad (22)$$

$$\begin{aligned} \hat{\eta} &= -\frac{\ddot{\phi}}{\hat{H}\dot{\phi}} = \frac{m_{pl}^2}{8\pi} \frac{\hat{V}_{,\phi\phi}}{\hat{V}} - \frac{m_{pl}^2}{16\pi} \left( \frac{\hat{V}_{,\phi}}{\hat{V}} \right)^2 \\ &\equiv \hat{\eta}_V - \hat{\epsilon}_V. \end{aligned} \quad (23)$$

We have defined another useful slow-roll parameter  $\hat{\delta}$  as [20]

$$\hat{\delta} = \frac{\dot{\hat{Q}}}{2\hat{H}\hat{Q}}, \quad \hat{Q} = \left( \frac{\dot{\phi}}{\hat{H}} \right)^2. \quad (24)$$

Note that  $\hat{\chi}$  is the order of  $\hat{\epsilon}$  from the definition.

From the third order action, Eq. (21), the bispectrum for  $\hat{\zeta}$  becomes in the interacting picture [8]

$$\langle \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \hat{\zeta}_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^3(\sum \mathbf{k}_i) \frac{(8\pi)^2}{m_{pl}^4} \frac{\hat{H}^4}{\hat{\epsilon}} \frac{1}{\Pi_i(2k_i^3)} \mathcal{A}, \quad (25)$$

where

$$\begin{aligned} \mathcal{A} &= 2 \frac{\ddot{\phi}}{\dot{\phi}\hat{H}} \sum_i k_i^3 + \frac{\dot{\phi}^2}{\hat{H}^2} \left[ \frac{1}{2} \sum_i k_i^3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 \right. \\ &\quad \left. + \frac{4}{k_{tot}} \sum_{i > j} k_i^2 k_j^2 \right], \end{aligned} \quad (26)$$

$$k_{tot} = k_1 + k_2 + k_3. \quad (27)$$

Then, we can read off the nonlinear parameter  $\hat{f}_{NL}$  from the relations om Eq. (9):

$$\hat{f}_{NL} \simeq \frac{5}{6}(\hat{\eta} - 2\hat{\epsilon}) = -\frac{5}{6}(\hat{\delta} + \hat{\epsilon}). \quad (28)$$

If we get back to the original Jordan frame, then from the fact that  $\hat{f}_{NL} = f_{NL}$ ,

$$\hat{f}_{NL} = -\frac{5}{6}(\hat{\delta} + \hat{\epsilon}) = f_{NL}. \quad (29)$$

We can easily find the relations of the slow-roll parameters between the Einstein frame and the original Jordan frame:

$$\hat{\epsilon} = \frac{\epsilon + \beta}{1 + \beta} - \frac{\dot{\beta}}{H(1 + \beta)^2}, \quad (30)$$

$$\hat{\delta} = \frac{\delta - \beta}{1 + \beta}, \quad (31)$$

where

$$\hat{Q} = \frac{Q}{\Omega^2}, \quad Q \equiv \frac{\dot{\phi}^2 + 6\dot{\Omega}^2}{(H + \dot{\Omega}/\Omega)^2},$$

$$\beta = \frac{\dot{\Omega}}{H\Omega}. \quad (32)$$

In the slow-roll approximation,  $\dot{\beta} \simeq 0$  [20], and for nonminimally coupled scalar field theory,

$$1 + \beta = 1 - \frac{\kappa\xi\phi}{H\Omega^2}\dot{\phi}. \quad (33)$$

Then,

$$f_{NL} = -\frac{5}{6} \frac{1}{1 + \beta} (\delta + \epsilon). \quad (34)$$

The nonminimal coupling constant  $\xi$  is restricted to a range of  $|\xi| < 10^{-3}$  [21] to have the number of  $e$ -folds needed for sufficient inflation [22]. With this constraint on  $\xi$ , the effect of nonminimal coupling,  $1/(1 + \beta)$ , does not contribute much to  $f_{NL}$  compared to the minimal coupling case. Thus, as long as  $\epsilon, \delta \ll 1$ , non-Gaussian signal of the single field slow-roll inflation model in a nonminimally coupled scalar field theory is difficult to observe by using Cosmic Microwave Background (CMB) experiments as in the Einstein gravity.

## V. CONCLUSIONS

We have considered non-Gaussianity when the scalar field is nonminimally coupled to gravity. It is convenient to deal with the equations of motion in a Einstein frame in which the scalar fields are minimally coupled to the gravity. The action in the nonminimally coupled scalar field theory is transformed into the Einstein gravity with canonical kinetic terms by using a suitable conformal transformation.

In linear perturbation theory, the gauge invariant quantity  $\zeta$ , which is constant on uniform energy density hypersurfaces for superhorizon scales, or  $\mathcal{R}$  on comoving hypersurfaces is

well known to be invariant under the conformal transformation [17, 18]. We have found that the gauge invariant quantity in nonlinear perturbation theory  $\zeta_i$  is also invariant under the conformal transformation. Conformal invariance of  $\zeta_i$  proves that the power spectrum and bispectrum of  $\zeta_i$  are also conformally invariant. These facts reveal that the nonlinear parameters  $f_{NL}$  in the two frames should be the same. Therefore, if we use Maldacena's result [8] in the Einstein frame, we can calculate the non-Gaussianity in nonminimally coupled scalar field theory via a conformal transformation.

For  $|\xi| < 10^{-3}$ , which is needed to ensure a sufficient number of  $e$ -folds, the effect of nonminimal coupling does not contribute much to the nonlinear parameter  $f_{NL}$  compared to the minimally coupling case. As in the single field inflation model in Einstein gravity, a non-Gaussian signal on the temperature anisotropy in nonminimally coupled scalar field theory will be difficult to observe in future CMB experiments.

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